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# Response of non-equilibrium systems with long-range initial correlations

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## Abstract

The long-time dynamics of the  $d$ -dimensional spherical model with a non-conserved order parameter and quenched from an initial state with long-range correlations is studied through the exact calculation of the two-time autocorrelation and autoresponse functions. In the ageing regime, these are given in terms of non-trivial universal scaling functions of both time variables. At criticality, five distinct types of ageing are found, depending on the form of the initial correlations, while at low temperatures only a single type of ageing exists. The autocorrelation and autoresponse exponents are shown to be generically different and to depend on the initial conditions. The scaling form of the two-time response functions agrees with a recent prediction coming from conformal invariance.

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## 1. Introduction

The slow dynamics of non-equilibrium systems displays several characteristic features which are absent for systems in thermodynamic equilibrium (see [1–4] for review). Most notable among these are the breakdown of the fluctuation–dissipation theorem and the ageing of the system as conveniently displayed in the two-time autocorrelation and autoresponse functions. While these features were first observed in glassy systems, they also occur in simple ferromagnetic spin systems without disorder. The present paper studies such relatively simple non-equilibrium systems.

Usually, the system is prepared in some initial state (a completely disordered initial state of effective infinite temperature is common) and then brought out of equilibrium, even at very long times, by a rapid quench to some final temperature  $T$  which may be below or equal to the critical temperature  $T_c$ . The subsequent evolution then takes place at that fixed temperature  $T$ . The main observables include the two-time autocorrelation function  $C(t, s)$

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and the autoresponse function  $R(t, s)$  where  $s$  is the waiting time and  $t$  the observation time. A convenient way to characterize the distance of the system from equilibrium is through the fluctuation–dissipation ratio [2, 5, 6]

$$X(t, s) = TR(t, s) \left( \frac{\partial C(t, s)}{\partial s} \right)^{-1}. \quad (1.1)$$

At equilibrium, both  $C$  and  $R$  depend only on the time difference  $\tau = t - s$  and  $X = X(\tau) = 1$ . Physically, this is realized if the initial quench was made to some temperature  $T > T_c$ . Then for timescales large as compared to the (finite) characteristic timescale  $\tau_{\text{eq}}$ , the system relaxes exponentially fast towards equilibrium.

On the other hand, if either  $T < T_c$  or  $T = T_c$ , an infinite spin system does not reach equilibrium on some finite timescale but instead undergoes either phase-ordering kinetics or non-equilibrium critical dynamics. In both cases, two-time observables such as  $C = C(t, s)$  and  $R = R(t, s)$  depend on *both* the waiting time  $s$  and the observation time  $t$  and not merely on their difference  $\tau = t - s$ . This breaking<sup>2</sup> of time translation invariance is usually referred to as *ageing* and will be used in this sense from now on. Consequently, the fluctuation–dissipation ratio  $X = X(t, s) \neq 1$  also becomes a non-trivial function of both  $s$  and  $t$ . Empirically, it is well established [1] that the ageing process is associated with dynamical scaling, that is in the scaling regime with times  $1 \ll s, \tau$  long enough after the initial quench, one finds

$$C(t, s) \sim s^{-b} f_C(t/s) \quad R(t, s) \sim s^{-1-a} f_R(t/s) \quad (1.2)$$

where  $a, b$  are non-equilibrium critical exponents and  $f_C$  and  $f_R$  are scaling functions. Here and in the following we always have  $t > s$ . For large arguments  $x = t/s \gg 1$ , these scaling functions typically behave as

$$f_C(x) \sim x^{-\lambda_C/z} \quad f_R(x) \sim x^{-\lambda_R/z} \quad (1.3)$$

where  $z$  is the dynamical exponent and  $\lambda_C, \lambda_R$  are the autocorrelation [8, 9] and autoresponse exponents<sup>3</sup>. Throughout this paper, we are only interested in this late-time regime where scaling occurs.

Usually, the long-time behaviour after a quench from a completely disordered state is studied. In this case, the available evidence as reviewed in [4], and based on results notably from the Glauber–Ising model [10, 11] and the spherical model with a non-conserved order parameter [12, 13] is consistent with the autocorrelation and autoresponse exponents being equal,  $\lambda_C = \lambda_R$ , and, for a quench exactly onto criticality  $T = T_c$ , with the additional relation  $a = b$ . However, exceptions to this rule exist. Consider the 2D XY model with a fully ordered zero-temperature initial state and quenched ‘upwards’ to some temperature  $T < T_{\text{KT}}$ , where  $T_{\text{KT}}$  is the Kosterlitz–Thouless transition temperature. A recent calculation by Berthier, *et al* [14] based on the spin-wave approximation has produced analytical results for both the autocorrelation and autoresponse scaling functions  $f_C(x)$  and  $f_R(x)$  and found the scaling ‘in complete agreement with the results obtained by Godrèche and Luck [13] for the spherical model at the critical point’ [14, p 1809]. Indeed one has  $a = b = \eta/2$ , where  $\eta = \eta(T)$  is the usual static (and temperature-dependent) critical exponent, but the autocorrelation exponent  $\lambda_C/z = \eta/4$  and the autoresponse exponent  $\lambda_R/z = 1 + \eta/4$  [14] are different from each other. We therefore ask ourselves, what may generically become of the relation between  $\lambda_C$  and  $\lambda_R$  for more general initial states than for fully disordered or fully ordered ones. Beyond a case study in a given model, this allows a test of the generic scaling and universality properties as

<sup>2</sup> This is associated with a breaking of the fluctuation–dissipation theorem. The timescale on which this breaking occurs has been studied very thoroughly in [7].

<sup>3</sup> The values of the exponents  $\lambda_C, \lambda_R$  (and also  $a, b$ ) depend on whether  $T < T_c$  or  $T = T_c$ , but we shall use the same notation in both cases.

reviewed above and should provide useful insight. The role of the initial conditions on ageing in spin-glasses has been and continues to be actively studied (see [12, 15] and references therein).

Questions of this sort are best addressed first in some non-trivial exactly solvable model before being studied further through simulations. We shall therefore examine for the kinetic spherical model described by a non-conserved Langevin equation (see section 2 for precise definitions) the role of long-range correlations in the initial state which are characterized by a power law for the Fourier transformation  $\tilde{C}(\mathbf{q})$  of the spin–spin correlator, in the low-momentum limit  $|\mathbf{q}| \rightarrow 0$ ,

$$\tilde{C}(\mathbf{q}) \sim |\mathbf{q}|^\alpha \quad (1.4)$$

and we shall study how the above scaling forms are affected, if at all, by varying  $\alpha$ . We can therefore interpolate between a disordered initial state with  $\alpha = 0$  and a fully ordered state ( $\alpha = -d$ ) as considered in [14]. While much is already known for  $T < T_c$  [16], we shall see that for  $T = T_c$  there occur several new regimes for the long-time ageing behaviour which depend on the value of  $\alpha$ . Our calculations suggest the new scaling relation  $\lambda_C = \lambda_R + \alpha_{\text{eff}}$ , where  $\alpha_{\text{eff}} = \alpha_{\text{eff}}(\alpha)$  is the effective value of  $\alpha$  which actually describes the long-time behaviour. We shall show that this also explains the results obtained in the 2D XY model [14] which are referred above.

Furthermore, introducing a new parameter into the kinetics permits an instructive test of the assertion by Godrèche and Luck [13] that the limit fluctuation–dissipation ratio

$$X_\infty = \lim_{s \rightarrow \infty} \lim_{t \rightarrow \infty} X(t, s) \quad (1.5)$$

is a universal number and we shall indeed confirm this universality (see section 3).

Finally, we recall that recently the precise form of the autoresponse function  $R(t, s)$  was derived from conformal invariance [17], with  $t, s \gg 1$  being inside the scaling regime [18, 19]:

$$R(t, s) = r_0 \left( \frac{t}{s} \right)^{1+a-\lambda_R/z} (t-s)^{-1-a} \quad (1.6)$$

where  $r_0$  is a normalization constant. As we shall see in section 4, this functional form is indeed recovered, but the exponents  $a$  and  $\lambda_R$  will be functions of  $\alpha$ .

The structure of this paper is as follows. In the next section, to give a self-contained presentation, we recall the main steps of the general formalism for the exact calculation of correlation and response functions. In section 3, we analyse these in the scaling limit where ageing occurs and find the scaling functions  $f_C(x)$  and  $f_R(x)$  as functions of the initial conditions characterized by the parameter  $\alpha$ . In section 4, the physical conclusions from these calculations are drawn. In the appendix, we shall briefly present some results on initial conditions in the 1D Glauber–Ising model such that the mean magnetization is non-vanishing.

## 2. Formalism

We now describe the calculation of the two-time autocorrelation and autoresponse functions in the exactly solvable spherical model in  $d$  spatial dimensions, for arbitrary initial conditions. Our calculation follows closely the standard lines established several times in the past (see [16, 20–24] for continuum field theories and [7, 12, 13, 25, 26] for lattice models). The effects of specific initial conditions on the long-time behaviour and ageing will be analysed in the next section.

The spherical model is defined in terms of real spin variables  $S_r$  attached to the sites of a  $d$ -dimensional hypercubic lattice and subject to the constraint

$$\sum_r S_r^2 = \mathcal{N} \quad (2.1)$$

where  $\mathcal{N}$  is the total number of sites and the usual spin Hamiltonian  $\mathcal{H} = -\sum_{\langle r,r' \rangle} S_r S_{r'}$  where the sum extends over the nearest neighbour pairs only. The (non-conserved) dynamics is given by the stochastic Langevin equation:

$$\frac{dS_r}{dt} = \sum_{s(r)} S_s - (2d + \mathfrak{z}(t))S_r + \eta_r(t) \quad (2.2)$$

where  $s(r)$  are the nearest neighbour sites of the site  $r$ , the Gaussian white noise  $\eta_r(t)$  has the correlation

$$\langle \eta_r(t) \eta_{r'}(t') \rangle = 2T \delta_{r,r'} \delta(t - t') \quad (2.3)$$

and  $\mathfrak{z}(t)$  is determined by satisfying the spherical constraint (2.1) in the mean. By a Fourier transformation

$$\tilde{f}(\mathbf{q}) = \sum_r f_r e^{-i\mathbf{q}\cdot\mathbf{r}} \quad f_r = (2\pi)^{-d} \int_{\mathcal{B}} d\mathbf{q} \tilde{f}(\mathbf{q}) e^{i\mathbf{q}\cdot\mathbf{r}} \quad (2.4)$$

where the integral is over the first Brillouin zone  $\mathcal{B}$ , equation (2.2) is transformed into

$$\frac{\partial \tilde{S}(\mathbf{q}, t)}{\partial t} = -[\omega(\mathbf{q}) + \mathfrak{z}(t)] \tilde{S}(\mathbf{q}, t) + \tilde{\eta}(\mathbf{q}, t) \quad (2.5)$$

where in addition, together with the  $|\mathbf{q}| \rightarrow 0$  limit

$$\omega(\mathbf{q}) = 2 \sum_{i=1}^d (1 - \cos q_i) \simeq q^2 \quad \langle \tilde{\eta}(\mathbf{q}, t) \tilde{\eta}(\mathbf{q}', t') \rangle = 2T (2\pi)^d \delta^d(\mathbf{q} + \mathbf{q}') \delta(t - t'). \quad (2.6)$$

The formal solution is [12, 13]

$$\begin{aligned} \tilde{S}(\mathbf{q}, t) &= \frac{\exp(-\omega(\mathbf{q})t)}{\sqrt{g(T, t)}} \left[ \tilde{S}(\mathbf{q}, 0) + \int_0^t dt' e^{\omega(\mathbf{q})t'} \sqrt{g(T, t')} \tilde{\eta}(\mathbf{q}, t') \right] \\ g(T, t) &= \exp\left(2 \int_0^t dt' \mathfrak{z}(t')\right) \end{aligned} \quad (2.7)$$

which forms the basis for all subsequent calculations.

The Lagrange multiplier  $g(T, t)$  is determined from the spherical constraint and the initial conditions. To see this, consider the equal-time spin–spin correlation function

$$C_{r-r'}(t) = \langle S_r(t) S_{r'}(t) \rangle \quad (2.8)$$

where spatial translation invariance is already taken into account. Here and in the following, the brackets denote the average over the ensemble of the initial conditions and over the thermal histories, i.e. the realizations of the noise  $\eta_r(t)$ . The spherical constraint (2.1) implies that

$$C_0(t) = \langle S_r(t) S_r(t) \rangle = 1. \quad (2.9)$$

In Fourier space the equal-time correlator  $\tilde{C}(\mathbf{q}, t)$  is obtained from

$$\langle \tilde{S}(\mathbf{q}, t) \tilde{S}(\mathbf{q}', t) \rangle = (2\pi)^d \delta^d(\mathbf{q} + \mathbf{q}') \tilde{C}(\mathbf{q}, t) \quad (2.10)$$

and is given by

$$\tilde{C}(\mathbf{q}, t) = \frac{\exp(-2\omega(\mathbf{q})t)}{g(T, t)} \left[ \tilde{C}(\mathbf{q}, 0) + 2T \int_0^t dt' e^{2\omega(\mathbf{q})t'} g(T, t') \right]. \quad (2.11)$$

From the spherical constraint (2.9), we have in Fourier space that  $\int d\mathbf{q} (2\pi)^{-d} \tilde{C}(\mathbf{q}, t) = 1$ , and this leads to the following Volterra integral equation for  $g(T, t)$  (see [12, 13]),

$$g(T, t) = A(t) + 2T \int_0^t dt' f(t-t') g(T, t') \quad (2.12)$$

where the two auxiliary functions  $f(t)$  and  $A(t)$  are defined as follows:

$$f(t) = \frac{1}{(2\pi)^d} \int d\mathbf{q} e^{-2\omega(\mathbf{q})t} = (e^{-4t} I_0(4t))^d \quad A(t) = \frac{1}{(2\pi)^d} \int d\mathbf{q} e^{-2\omega(\mathbf{q})t} \tilde{C}(\mathbf{q}, 0) \quad (2.13)$$

and where  $I_0$  is a modified Bessel function [27]. The solution of equation (2.12) is found from a Laplace transformation:

$$\bar{f}(p) = \int_0^\infty dt f(t) e^{-pt} \quad (2.14)$$

and is given by

$$\bar{g}(T, p) = \frac{\bar{A}(p)}{1 - 2T\bar{f}(p)}. \quad (2.15)$$

Therefore the entire evolution of the system, starting from a freely chosen initial condition, can be described in terms of the properties of the functions  $f(t)$  and  $A(t)$ . In particular, the initial state is characterized exclusively by the initial correlator  $\tilde{C}(\mathbf{q}, 0)$  and these data enter explicitly only into the function  $A(t)$ . The simplest case may be considered to be an initial state without any correlations. Then  $C_r(0) = \delta_{r,0}$ , therefore  $\tilde{C}(\mathbf{q}, 0) = 1$  and thus  $A(t) = f(t)$ . This case has been analysed in great detail (see [7, 12, 26] and in particular [13]).

We postpone the analysis of the effects of initial conditions given by  $\tilde{C}(\mathbf{q}, 0)$  to the next section and now give the general expressions for the two-time correlators and the two-time response functions. The two-time correlation function is defined as

$$C_{r-r'}(t, s) = \langle S_r(t) S_{r'}(s) \rangle \quad (2.16)$$

where  $s$  is the waiting time,  $t$  the observation time and  $t \geq s \geq 0$  always. In Fourier space, it is easy to see that

$$\tilde{C}(\mathbf{q}, t, s) = \tilde{C}(\mathbf{q}, s) e^{-\omega(\mathbf{q})(t-s)} \sqrt{\frac{g(T, s)}{g(T, t)}} \quad (2.17)$$

and where expression (2.11) for the single-time correlator has been used. Below, we shall be mainly interested in the two-time autocorrelation:

$$\begin{aligned} C(t, s) = C_0(t, s) &= (2\pi)^{-d} \int d\mathbf{q} \tilde{C}(\mathbf{q}, t, s) \\ &= \frac{1}{\sqrt{g(T, t)g(T, s)}} \left[ A\left(\frac{t+s}{2}\right) + 2T \int_0^s ds' f\left(\frac{t+s}{2} - s'\right) g(T, s') \right] \end{aligned} \quad (2.18)$$

which we shall analyse in the next section. The response function is obtained in the usual way [12, 13, 21, 22] by adding a small magnetic field term  $\delta\mathcal{H} = -\sum_r h_r(t) S_r(t)$  to the Hamiltonian. This leads to an extra term  $h_r(t)$  on the right-hand side of the Langevin equation (2.2). Provided that causality and spatial translation invariance hold, we have to first order

$$\langle S_r(t) \rangle = \int_0^t ds \sum_{r'} R_{r-r'}(t, s) h_{r'}(s) + \dots \quad (2.19)$$

which defines the (linear) response function  $R_r(t, s)$ . The calculation is completely standard [12, 13, 21] and we merely quote the result

$$\tilde{R}(\mathbf{q}, t, s) = \left. \frac{\delta \langle \tilde{S}(\mathbf{q}, t) \rangle}{\delta \tilde{h}(\mathbf{q}, s)} \right|_{h_r=0} = e^{-\omega(\mathbf{q})(t-s)} \sqrt{\frac{g(T, s)}{g(T, t)}}. \quad (2.20)$$

Again, we shall be mainly concerned with the autoresponse function

$$R(t, s) = R_0(t, s) = (2\pi)^{-d} \int d\mathbf{q} \tilde{R}(\mathbf{q}, t, s) = f\left(\frac{t-s}{2}\right) \sqrt{\frac{g(T, s)}{g(T, t)}} \quad (2.21)$$

Equations (2.11), (2.18) and (2.21) give the single-time correlation function and two-time autocorrelation and autoresponse functions, respectively, for yet arbitrary initial conditions. Together with the spherical constraint (2.12), which fixes the function  $g(T, t)$  in terms of the auxiliary functions  $f(t)$  and  $A(t)$ , these are the main results of this section.

### 3. Analysis

Our aim is to consider the effects of long-range correlations in the initial state on the long-time and ageing behaviours of the model. For the long-time behaviour, only the low-momentum regime should be relevant, which we take to be of the form [16]

$$\tilde{C}(\mathbf{q}, 0) = c_0 + c_\alpha |\mathbf{q}|^\alpha \quad (3.1)$$

where  $\alpha$  is a free parameter and  $c_0, c_\alpha$  are normalization constants. For  $\alpha > 0$ , the long-time behaviour depends, to leading order, only on the first term while for  $\alpha < 0$ , we have effectively  $c_0 = 0$ , to leading order. For the purpose of this paper, namely the study of the role of long-range correlations in the initial conditions as parametrized by  $\alpha$ , it is sufficient to study the case  $c_0 = 0$ , which we shall assume from now on. For  $\alpha < 0$ , this corresponds to initial correlations of the form  $C_r(0) \sim r^{-d-\alpha}$  for large distances  $r = |\mathbf{r}|$ . This initial condition should lead to the following  $t \rightarrow \infty$  asymptotics of the auxiliary function:

$$A(t) \simeq a_\alpha t^{-(d+\alpha)/2} \quad a_\alpha = c_\alpha (2\pi)^{-d} \int d\mathbf{u} e^{-2u^2} u^\alpha. \quad (3.2)$$

The case without initial correlations considered previously [7, 12, 13, 26] corresponds to  $\tilde{C}(\mathbf{q}, 0) = 1$  and is recovered if we formally set  $\alpha = 0$ . Any differences between the scaling behaviour coming from the initial conditions (3.1) and for those analysed previously must relate to a different long-time behaviour of the functions  $A(t)$  and  $f(t)$  (it is well known that for large  $t$ , one has  $f(t) \simeq (8\pi t)^{-d/2}$ ). The purpose of this section is to carry through the technical analysis and especially to obtain the explicit scaling functions for the autocorrelation, autoresponse and the fluctuation–dissipation ratio. The physical discussion of the results will be presented in section 4.

The single-time correlator  $C_r(t)$  has been analysed in great detail by Coniglio *et al* [23] in the context of coarse-grained field theory.

We now analyse the long-time behaviour of the two-time correlators and response functions. For the questions we are interested in, namely the breaking of the fluctuation–dissipation theorem and/or ageing effects, it is enough to restrict to temperatures at or below criticality. Indeed, for  $T > T_c$ , the system will simply relax to equilibrium within a finite timescale  $\tau_{\text{eq}} \sim (T - T_c)^{-2\nu}$ , where  $\nu$  is the known equilibrium correlation length exponent (see [13] and the references therein). For the same reason, we restrict to  $d > 2$  throughout such that there is always a phase transition at a non-vanishing critical temperature  $T_c > 0$ . In addition, the early stages of the evolution of the system (notably the evolution of the

fluctuation–dissipation ratio) will depend on whether the system was initially prepared in an equilibrium state or not but this information requires more knowledge on the initial state than merely its low-momentum behaviour equation (3.1). Rather, the behaviour we are interested in is described by the scaling or ageing limit which is reached when both the waiting time  $s$  and the observation time  $t$  become simultaneously large, that is  $1 \ll s \sim t - s$ . As we shall be interested in the scaling properties of  $C(t, s)$  and  $R(t, s)$ , it is useful to work with the dimensionless scaling variable

$$x = \frac{t}{s} > 1. \quad (3.3)$$

Taking the scaling limit  $t, s \rightarrow \infty$  with  $x$  fixed, it is convenient to rewrite the results (2.18) and (2.21) of the previous section as follows:

$$C(t, s) \simeq \frac{s^{-d/2+1}}{\sqrt{g(T, s)g(T, t)}} \left[ a_\alpha s^{-\alpha/2-1} \left( \frac{x+1}{2} \right)^{-\frac{1}{2}(d+\alpha)} + \frac{2T}{(4\pi)^{d/2}} \int_0^1 d\theta g(T, s\theta) (x+1-2\theta)^{-d/2} \right] \quad (3.4)$$

$$R(t, s) \simeq (4\pi)^{-d/2} s^{-d/2} (x-1)^{-d/2} \sqrt{\frac{g(T, s)}{g(T, t)}}$$

where we have neglected the correction terms to the leading scaling behaviour. Next, we need the leading behaviour of  $g(T, t)$  for large  $t$ .

### 3.1. Non-equilibrium critical dynamics

We first consider the case when  $T = T_c$ , the equilibrium critical temperature. In general, we expect the leading asymptotics of  $g(T, t)$  to be of the form

$$g(T, t) \simeq g_0(\alpha, d) t^{\psi(\alpha, d)} \quad t \gg 1 \quad (3.5)$$

where the dependence of  $\psi = \psi(\alpha, d)$  must be calculated and  $g_0$  is a constant. We determine  $\psi$  by finding first from equation (2.15) the low- $p$  behaviour of  $\bar{g}(T, p)$ , which in turn is given by the small- $p$  behaviours of  $\bar{f}(p)$  and  $\bar{A}(p)$ . These may be found from the integral representations (2.14) using well-established techniques [13, 28]. At the critical point, the results are as follows:

$$\begin{aligned} \bar{f}(p) &\simeq A_1 + \frac{\Gamma(1-d/2)}{(8\pi)^{d/2}} p^{d/2-1} + \dots & 2 < d < 4 \\ \bar{f}(p) &\simeq A_1 - A_2 p + \dots & d > 4 \end{aligned} \quad (3.6)$$

where  $T_c = 1/(2A_1)$  is the equilibrium critical temperature and  $A_k = (2\pi)^{-d} \int d\mathbf{q} (2\omega(\mathbf{q}))^{-k}$ ,  $k = 1, 2, \dots$ , which exist for  $d > 2k$ . Similarly, we find

$$\begin{aligned} \bar{A}(p) &\simeq a_\alpha \Gamma\left(1 - \frac{1}{2}(d+\alpha)\right) p^{-1+(d+\alpha)/2} & 0 < d+\alpha < 2 \\ \bar{A}(p) &\simeq B_1 + O(p, p^{-1+(d+\alpha)/2}) & d+\alpha > 2 \end{aligned} \quad (3.7)$$

where  $a_\alpha$  is given in equation (3.2) and  $B_1 = (2\pi)^{-d} \int d\mathbf{q} \tilde{C}(\mathbf{q}, 0) (2\omega(\mathbf{q}))^{-1}$  which exists for  $d+\alpha > 2$  (if either  $d = 2, 4$  or  $d+\alpha = 0, 2$ , additional logarithmic factors will be present, which we shall discard throughout). From this,  $\bar{g}(T, p)$  follows and transforming back to  $g(T, t)$ , the exponent  $\psi(\alpha, d)$  describing the leading behaviour for large  $t$  is found. We summarize the results in table 1 and identify five regimes where the behaviour of  $g(T, t)$  is different. It is convenient to characterize these regimes in terms of an effective dimension

$$D = d + \alpha + 2 \quad (3.8)$$



**Table 1.** Values of the exponent  $\psi = \psi(\alpha, d)$  which describes the long-time behaviour of  $g(T_c, t)$  according to equation (3.5) and the exponent  $\varphi = 1 + \psi + \alpha/2$  in the five different scaling regimes at criticality.

Regime	Conditions		$\psi$	$\varphi$
I	$2 < d < 4$	$2 < D < 4$	$-1 - \alpha/2$	0
II	$4 < d$	$2 < D < 4$	$1 - (d + \alpha)/2$	$(4 - d)/2$
III	$2 < d < 4$	$4 < D$	$d/2 - 2$	$(d + \alpha)/2 - 1$
IV	$4 < d$	$4 < D$	$\alpha > -2$	0
V	$4 < d$	$4 < D$	$\alpha < -2$	0

**Table 2.** Values of the critical autocorrelation and autoresponse exponents  $a, b, \lambda_C$  and  $\lambda_R$  as defined in equations (3.13) and (3.23) in the five scaling regimes.

Regime	$a$	$b$	$\lambda_C$	$\lambda_R$
I	$d/2 - 1$	$d/2 - 1$	$d + \alpha/2 - 1$	$d - \alpha/2 - 1$
II	$d/2 - 1$	1	$(d + \alpha)/2 + 1$	$(d - \alpha)/2 + 1$
III	$d/2 - 1$	$d/2 - 1$	$3d/2 - 2$	$3d/2 - 2$
IV	$d/2 - 1$	$d/2 - 1$	$d$	$d$
V	$d/2 - 1$	$d/2 + \alpha/2$	$d + \alpha$	$d$

the meaning of which we shall discuss in section 4. From these results, we find the following scaling forms:

$$\begin{aligned}
 C(t, s) &= (4\pi)^{-d/2} s^{-d/2+1} [s^{-\varphi} M_0(x) + K_0(x)] \\
 R(t, s) &= (4\pi)^{-d/2} s^{-d/2} (x - 1)^{-d/2} x^{-\psi/2}
 \end{aligned}
 \tag{3.9}$$

where the values of the exponent  $\varphi = 1 + \psi + \alpha/2$  are also listed in table 1 and

$$\begin{aligned}
 M_0(x) &= \frac{\alpha_\alpha (4\pi)^{d/2}}{g_0} x^{-\psi/2} \left(\frac{x+1}{2}\right)^{-(d+\alpha)/2} \\
 K_0(x) &= 2T x^{-\psi/2} \int_0^1 dw w^\psi (x+1-2w)^{-d/2}.
 \end{aligned}
 \tag{3.10}$$

Furthermore, the scaling of the fluctuation–dissipation ratio can be written in the form:

$$X(t, s) = T \frac{(x - 1)^{-d/2} x^{-\psi/2}}{s^{-\varphi} M(x) + K(x)}
 \tag{3.11}$$

where the functions  $M(x)$  and  $K(x)$  are defined by

$$\begin{aligned}
 M(x) &= -\left(\frac{d + \alpha}{2} + \psi\right) M_0(x) - x \frac{dM_0(x)}{dx} \\
 K(x) &= \left(1 - \frac{d}{2}\right) K_0(x) - x \frac{dK_0(x)}{dx}
 \end{aligned}
 \tag{3.12}$$

and we see that the scaling of  $C(t, s)$  and consequently of  $X(t, s)$  depends on the sign of  $\varphi$ . For that reason, the regimes IV and V have to be distinguished.

We can now list the scaling functions for both the autocorrelation and autoresponse functions

$$C(t, s) = (4\pi)^{-d/2} T_c s^{-b} f_C(x) \quad R(t, s) = (4\pi)^{-d/2} s^{-1-a} f_R(x)
 \tag{3.13}$$

together with the fluctuation–dissipation ratio. The results for the exponents  $a$  and  $b$  are given in table 2 below. For the response function, the scaling function simply is in all five regimes

$$f_R(x) = (x - 1)^{-d/2} x^{-\psi/2}
 \tag{3.14}$$

and the values of  $\psi$  can be read from table 1. We shall return to this result in section 4.

The functions  $f_C(x)$  and  $X(t, s)$  are listed below: for regime I, we find

$$f_C(x) = \left[ 2^{1+\alpha/2} \left| \frac{\Gamma(1-d/2)\Gamma(-\alpha/2)}{\Gamma(1-(d+\alpha)/2)} \right| x^{-(d+\alpha/2-1)/2} + 2x^{1/2+\alpha/4} \int_0^1 dy y^{-1-\alpha/2} (x+1-2y)^{-d/2} \right] \simeq 2^{1+\alpha/2} \left| \frac{\Gamma(1-d/2)\Gamma(-\alpha/2)}{\Gamma(1-(d+\alpha)/2)} \right| x^{-d/2-\alpha/4+1/2} \quad x \rightarrow \infty. \quad (3.15)$$

The fluctuation–dissipation ratio becomes, in the scaling limit, a function of  $x$  only, which reads for  $\alpha \neq -2$

$$X = X(x) = (x-1)^{-d/2} \left[ - \left| \frac{\Gamma(1-d/2)\Gamma(-\alpha/2)}{\Gamma(1-(d+\alpha)/2)} \right| 2^{\alpha/2} \left( \frac{d+\alpha}{x+1} - \frac{\alpha}{2} - 1 \right) x^{-(d+\alpha)/2} + 2 \int_0^1 dy y^{-1-\alpha/2} (x+1-2y)^{-d/2} \left( \frac{1}{2} - \frac{\alpha}{4} + \frac{d}{2} \frac{2y-1}{x+1-2y} \right) \right]^{-1} \simeq \left| \frac{\Gamma(1-d/2)\Gamma(-\alpha/2)}{\Gamma(1-(d+\alpha)/2)} \right| \frac{2}{2+\alpha} (2x)^{\alpha/2} \quad x \rightarrow \infty \quad (3.16)$$

together with the leading behaviour for infinitely separated timescales  $x \rightarrow \infty$ . For  $\alpha = -2$ , we find

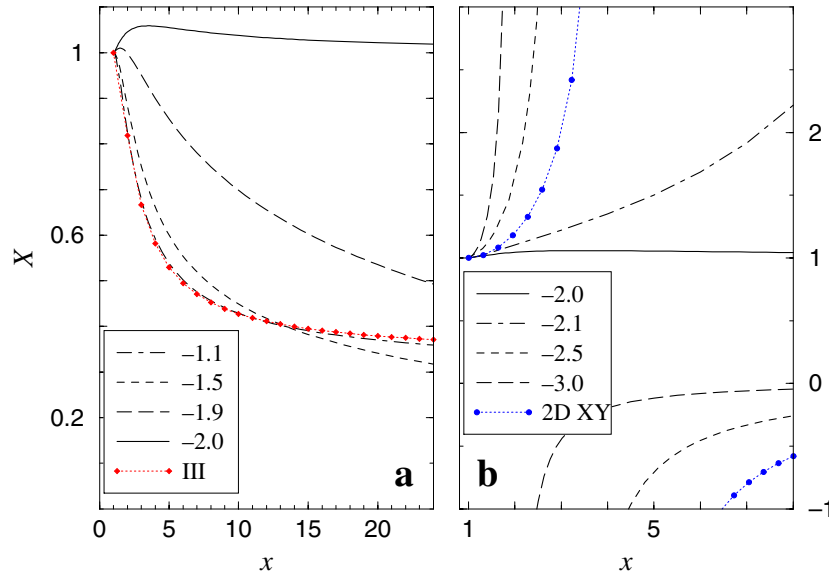
$$X = X(x) = \left[ 1 + \left( \frac{x-1}{x+1} \right)^{d/2} \left( 1 - \left( \frac{x}{x+1} \right)^{1-d/2} \right) \right]^{-1} \simeq 1 \quad x \rightarrow \infty. \quad (3.17)$$

Therefore, if  $\alpha \neq -2$ , the limit fluctuation–dissipation ratio has the universal value  $X_\infty = 0$  but the approach towards that limit does depend on the initial condition, while for  $\alpha = -2$ , we have  $X_\infty = 1$  signalling that an equilibrium state will be reached.

The behaviour of the scaling function  $X(x)$  is illustrated in figure 1. In the left part (figure 1(a)) we consider initial states with  $\alpha \geq -2$  which are more disordered than the critical equilibrium state. In all cases, one starts from equilibrium at equal times, since  $X(1) = 1$ . If  $\alpha \neq -2$ , the fluctuation–dissipation ratio decays with increasing  $x$ . However, if  $\alpha$  approaches the border between regions I and III, the scaling function  $X^{(I)}(x)$  obtained for region I goes over smoothly into that found for region III. Close to that boundary, quite large values of  $x$  are needed in order to distinguish these functions. On the other hand, for  $\alpha = -2$ , the long-range properties of the initial state are the same as for the equilibrium state. Although the system departs initially from equilibrium, since  $X(x) > 1$ , the non-equilibrium short-range correlations are successively equilibrated and the system finally arrives at the equilibrium value of the limit fluctuation–dissipation ratio  $X_\infty = 1$ . If we now consider the case  $\alpha \leq -2$  (figure 1(b)) with an initial state more ordered than the equilibrium state, the behaviour is completely different. Starting from  $X(1) = 1$ , the fluctuation–dissipation ratio increases with  $x$  and encounters a singularity at some finite value  $x_s = x_s(d, \alpha)$ . For  $x > x_s$ , it is negative and rapidly decays to zero with increasing  $x$ .

For regime II, we find

$$f_C(x) = 2(4\pi)^{d/2} A_2 \left( 1 - \frac{d+\alpha}{2} \right) x^{(d+\alpha)/4-1/2} \left( \frac{x+1}{2} \right)^{-(d+\alpha)/2} \\ X(x_s, s) = \frac{-(4\pi)^{-d/2} (x-1)^{-d/2}}{A_2(1-(d+\alpha)/2)(1+(d+\alpha)(3x+1)(2x+2)^{-1})} \left( \frac{x+1}{2} \right)^{(d+\alpha)/2} s^{-(d-4)/2} \quad (3.18)$$



**Figure 1.** The scaling of the fluctuation–dissipation ratio  $X(x)$  as a function of  $x = t/s$  in regime I for  $d = 3$  and for several values of  $\alpha$ . In panel (a), we also include the result  $X^{(III)}(x)$  found for regime III and given in equation (3.19). In panel (b) we also show the function  $X_{2DXY}(x)$  obtained in the 2D XY model in the spin-wave approximation [14] and given in equation (4.4).

where the integral  $A_2$  was defined above, after equation (3.6). Therefore for any value of  $x$ ,  $X(x, s) = 0$  in the scaling limit  $s \rightarrow \infty$ , since  $d > 4$  here.

For regime III, we have

$$f_C(x) = \frac{4}{d-2} \frac{(x-1)^{-d/2+1} x^{1-d/4}}{x+1} \quad X = X(x) = \left( 1 + \frac{2}{d-2} \left( \frac{x-1}{x+1} \right)^2 \right)^{-1} \quad (3.19)$$

which reproduces the scaling functions previously found [13] for  $\alpha = 0$ . The entire scaling functions turn out to be completely independent of  $\alpha$ . Therefore, the scaling behaviour in this region is governed by the effective value  $\alpha_{\text{eff}} = 0$  in the initial condition (3.1) and all scaling functions are universal. In particular, the limit fluctuation–dissipation ratio  $X_\infty = \lim_{x \rightarrow \infty} X(x) = 1 - 2/d$  is a universal number, as expected.

We have for the case of regime IV:

$$f_C(x) = \frac{2}{d-2} \left( (x-1)^{-d/2+1} - (x+1)^{-d/2+1} \right) \quad X = X(x) = \left( 1 + \left( \frac{x-1}{x+1} \right)^{d/2} \right)^{-1} \quad (3.20)$$

which again agrees completely with the earlier results found for  $\alpha = 0$  [13]. Again, the scaling functions are independent of  $\alpha$  and thus  $\alpha_{\text{eff}} = 0$ . In particular  $X_\infty = 1/2$  is a universal number.

Finally, in the regime V, we find

$$f_C(x) = 2(4\pi)^{d/2} A_2 B_V \left( \frac{x+1}{2} \right)^{-(d+\alpha)/2} \quad (3.21)$$

$$X(x, s) = \frac{-(4\pi)^{-d/2} (x-1)^{-d/2} (x+1)}{A_2 B_V (d+\alpha) (2x+1)} \left( \frac{x+1}{2} \right)^{(d+\alpha)/2} s^{1+\alpha/2}$$

where the constant  $B_V$  is given by

$$B_V = \left[ \int du u^\alpha e^{-2u^2} \right] \cdot \left[ \int du \frac{u^\alpha}{2\omega(u)} \right]^{-1}. \quad (3.22)$$

Therefore  $X(x, s, s) = 0$  in the scaling limit, since  $1 + \alpha/2 < 0$ .

From the  $x \rightarrow \infty$  asymptotics

$$f_C(x) \sim x^{-\lambda_C/z} \quad f_R(x) \sim x^{-\lambda_R/z} \quad (3.23)$$

and the known fact that  $z = 2$  [1] for the spherical model at and below  $T_c$ , we read the critical autocorrelation and autoresponse exponents  $\lambda_C$  and  $\lambda_R$  and summarize the results in table 2. They will be discussed in section 4.

### 3.2. Phase ordering

Having found the scaling behaviour at criticality, we now turn to the ordered phase, where  $T < T_c$ . This case was considered long ago in the context of the coarse-grained  $O(n)$ -symmetric field theory in the limit  $n \rightarrow \infty$  [16, 21]. The relations (3.4) remain valid, but the Lagrange multiplier  $g(T, t)$  now has for large times, the leading behaviour

$$g(T, t) \simeq \frac{\alpha_\alpha}{M_{\text{eq}}^2} t^{-(d+\alpha)/2} \quad (3.24)$$

where  $M_{\text{eq}}^2 = 1 - T/T_c$  is the squared equilibrium magnetization. In the scaling limit  $s \rightarrow \infty, t \rightarrow \infty$  but  $x = t/s$  fixed, one has for  $\alpha \leq 0$

$$\begin{aligned} C(t, s) &= M_{\text{eq}}^2 \left( \frac{(x+1)^2}{4x} \right)^{-(d+\alpha)/4} \\ R(t, s) &= (4\pi)^{-d/2} s^{-d/2} (x-1)^{-d/2} x^{(d+\alpha)/4} \\ X(t, s) &= \frac{4^{1-(d+\alpha)/4} T}{(4\pi)^{d/2} (d+\alpha) M_{\text{eq}}^2} \frac{(x+1)^{d/2+\alpha+1}}{(x-1)^{d/2+1}} s^{-d/2+1}. \end{aligned} \quad (3.25)$$

For  $\alpha = 0$ , these results were already known [13, 21, 26] (for  $\alpha > 0$ , the constant term in (3.1) is dominating the long-time behaviour). As expected, the fluctuation–dissipation ratio  $X = 0$  in the scaling limit. The scaling functions  $f_C(x)$  and  $f_R(x)$  are defined as

$$C(t, s) = M_{\text{eq}}^2 f_C(x) \quad R(t, s) = (4\pi)^{-d/2} s^{-1-a} f_R(x) \quad (3.26)$$

and we obtain

$$a = \frac{d}{2} - 1 \quad f_C(x) = \left( \frac{(x+1)^2}{4x} \right)^{-(d+\alpha)/4} \quad f_R(x) = (x-1)^{-d/2} x^{(d+\alpha)/4} \quad (3.27)$$

and formally  $b = 0$  if we were to compare with the scaling (3.13) at criticality. From these, using again equation (3.23), we have the autocorrelation and autoresponse exponents

$$\lambda_C = \frac{1}{2}(d+\alpha) \quad \lambda_R = \frac{1}{2}(d-\alpha). \quad (3.28)$$

These results, valid for  $T < T_c$ , supplement those for the critical case  $T = T_c$  given in table 2. For arbitrary  $\alpha < 0$  the value of  $\lambda_C$  was already known<sup>4</sup> [16]. A long time ago, Newman and Bray [21] studied the case of zero waiting time,  $s = 0$ . From equations (2.18) and (2.21) and the initial condition  $g(T, 0) = 1$ , one has

$$C(t, 0) = \frac{A(t/2)}{\sqrt{g(T, t)}} \sim M_{\text{eq}} t^{-(d+\alpha)/4} \quad R(t, 0) = \frac{f(t/2)}{\sqrt{g(T, t)}} \sim M_{\text{eq}} t^{-(d-\alpha)/4} \quad (3.29)$$

in full agreement with their results.

<sup>4</sup> For the  $O(n)$  model with finite  $n$ ,  $\lambda_C = d/2$  for  $0 > \alpha > \alpha_c$ , where  $\alpha_c$  is known to leading order in  $1/n$ . The result of equation (3.28) for  $\lambda_C$  only holds true if  $\alpha < \alpha_c$ . In the  $n \rightarrow \infty$  limit,  $\alpha_c = 0$  [16].

#### 4. Discussion

The long-time behaviour of the kinetic spherical model depends on the spatial dimension  $d$  and the parameter  $\alpha$  which characterizes the initial condition, equation (3.1). We have already introduced the effective dimension  $D$  (see equation (3.8)). In terms of these, the equilibrium correlation function  $C_{\text{eq}}(\mathbf{r})$  at criticality and the initial correlation function  $C_{\text{ini}}(\mathbf{r}) = C_r(0)$  scale for large distances  $r = |\mathbf{r}|$  as

$$C_{\text{eq}}(\mathbf{r}) \sim r^{-(d-2)} \quad C_{\text{ini}}(\mathbf{r}) \sim r^{-(D-2)} \quad (4.1)$$

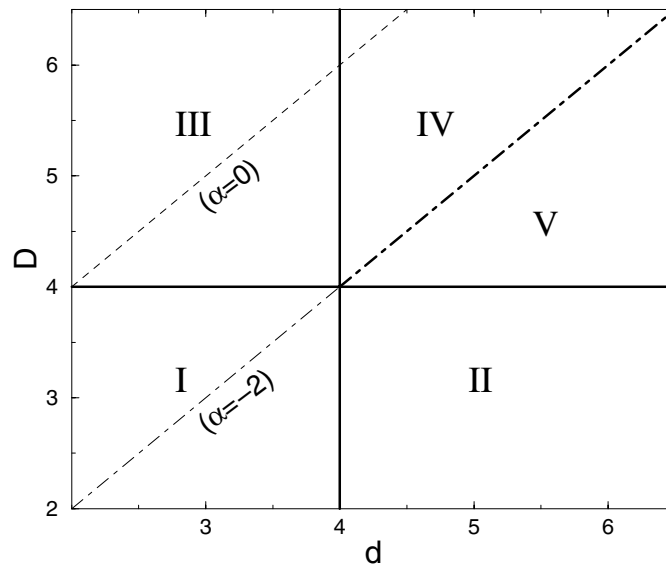
(since the equilibrium critical exponent  $\eta = 0$  for the spherical model). Therefore, we have studied the influence of the fluctuations of an effectively  $D$ -dimensional system on a model defined in  $d$  dimensions<sup>5</sup>. We can conclude:

1. In the scaling limit which describes the ageing of the system, the results obtained for a quench to a temperature  $T < T_c$  below the critical point and given in equations (3.27) and (3.28) show that the effect of initial long-range correlations persists for all times of the ongoing non-equilibrium behaviour, but without provoking any qualitative changes, as observed long ago [16, 21]. Remarkably enough, however, for times  $1 \ll t - s \ll s$  which in the scaling limit corresponds to  $x \simeq 1$  and for any initial condition, the two-time autocorrelations saturate at a plateau value  $C(x = 1) \simeq M_{\text{eq}}^2$  (see equation (3.25)). For larger differences between the waiting time  $s$  and the observation time  $t$ ,  $C(t, s)$  decays to zero according to a power law which now again depends on the initial conditions and the system shows thus a sort of memory of its initial state.

Surprisingly, the results for a quench precisely to  $T = T_c$  are qualitatively different. They are summarized in tables 1 and 2 and the resulting phase diagram is shown in figure 2. Indeed, we find five regimes with different ageing behaviours. These regimes are distinguished by the presence (when  $2 < d < 4$  and/or  $2 < D < 4$ ) of strong critical fluctuations in either the equilibrium or the initial state, respectively, or else their absence (when  $d > 4$  and/or  $D > 4$ ) when either the equilibrium or the initial state are in the mean-field regime. In addition, if both  $d > 4$  and  $D > 4$ , it is of relevance whether the initial correlations decay faster than those in the equilibrium state or not. If  $d = D$ , the system is prepared in its critical equilibrium state, the fluctuation–dissipation theorem is valid and no ageing occurs. That is the situation of *equilibrium* critical dynamics. Specifically, we find

- (a) Only if the initial correlations are in the mean-field regime, i.e.  $D > 4$ , and if in addition the initial correlations decay faster than in equilibrium, the system may show a non-vanishing value of the fluctuation–dissipation ratio  $X_\infty$ . In these cases (regimes III/IV), the entire scaling functions  $f_C(x)$ ,  $f_R(x)$  and  $X(x)$  do *not* depend on the initial conditions at all and agree with the previously known results obtained for a disordered initial state of infinite temperature (see [12, 13, 26]). One therefore has effective initial conditions such that  $\alpha_{\text{eff}} = 0$ . That finding is in full agreement with the expected [4, 13] universality of these scaling functions and of  $X_\infty$  in particular.
- (b) A non-trivial result for the scaling of the fluctuation–dissipation ratio  $X(x)$  is found if both the equilibrium and the initial states are fluctuation-dominated (regime I). The limit value for very large separations  $x = t/s \rightarrow \infty$  between the waiting time  $s$  and the observation time  $t$  is the universal value  $X_\infty = 0$ , provided that  $\alpha \neq -2$ , but the approach towards this limit depends on the initial condition through  $X(x) \sim x^{-|\alpha|/2}$ ,

<sup>5</sup> Practically, long-range initial conditions as studied here might be realized by coupling the degrees of freedom of the model under study to those of another system *at* criticality. Alternatively, one might consider a spin system with long-range interactions  $J = J(r)$  and use this to prepare an initial state with long-range correlations.



**Figure 2.** Kinetic phase diagram at  $T = T_c$  depending on the spatial dimension  $d$  and on the effective dimension  $D = d + \alpha + 2$  where  $\alpha$  is the exponent of the initial condition equation (3.1). The five different scaling regimes I, ..., V are shown. The dashed line ( $\alpha = 0$ ) corresponds to the case of a completely disordered initial state, the dashed-dotted line ( $\alpha = -2$ ) corresponds to an equilibrium initial state at  $T = T_c$  and the fully ordered zero-temperature initial state corresponds to the line  $D = 2$ .

as was illustrated in figure 1. It is not impossible that the rather trivial value  $X_\infty = 0$  which is more typical of a low-temperature phase might be a peculiarity of the spherical model. On the other hand, if  $\alpha = -2$ , then  $X_\infty = 1$  and the system evolves towards equilibrium.

- (c) If the equilibrium state is in the mean-field regime and if in addition the initial correlations decay slower than in equilibrium, the fluctuation–dissipation ratio  $X$  vanishes in the scaling limit, independently of the value of  $x = t/s$  (regimes II/V). Despite having fixed the temperature at its critical value, this behaviour is typical for a quench into the low-temperature ordered phase.

It would be of interest to see whether a similar variety of different types of non-equilibrium critical dynamics may be established for different spin systems (especially with  $z \neq 2$ ) and in particular, whether the role of the equilibrium and/or the initial state being in the mean-field regime can be confirmed. As a preparation for this we briefly treat in the appendix, an example where the mean magnetization does not vanish.

2. Turning to the values of the autocorrelation exponent  $\lambda_C$  and the autoresponse exponent  $\lambda_R$ , we find that one of the following two scenarios is realized: either (i) the two exponents are equal

$$\lambda_C = \lambda_R \quad (4.2)$$

which occurs at criticality in the regimes III and IV characterized by  $\alpha_{\text{eff}} = 0$  and where also  $X_\infty \neq 0$  or else (ii) they satisfy the relation

$$\lambda_C = \lambda_R + \alpha_{\text{eff}} \quad (4.3)$$

which is realized in the entire ordered phase and for the critical regimes I, II and V, where  $\alpha_{\text{eff}} = \alpha$  and  $X_\infty = 0$ . As an extreme case, this also includes a completely ordered initial

state, where  $C_{\text{ini}}(r) \sim \text{cste}$  which corresponds to  $\alpha = -d$ . Up to eventual logarithmic corrections, this case may be included by taking the limit  $D \rightarrow 2+$ .

We point out that the results  $\lambda_C/z = \eta/4$  and  $\lambda_R/z = 1 + \eta/4$  obtained for the 2D XY model [14] for a fully ordered initial state (therefore  $\alpha = -d = -2$ ) in the low-temperature phase are consistent with equation (4.3), since  $z = 2$  in that model. Indeed, the behaviour of the fluctuation–dissipation ratio  $X_{2\text{D XY}}(x)$  of the 2D XY model is quite analogous to that found here for the spherical model in regime I at criticality. In figure 1(b) we show the fluctuation–dissipation ratio of the 2D XY model when starting from a fully ordered initial state:

$$X_{2\text{D XY}}(x) = \left[ 1 - \frac{(x-1)^2}{2(x+1)} \right]^{-1} \quad (4.4)$$

which is valid in the entire low-temperature phase, using the spin-wave approximation [14]. The similarity with the functions  $X(x)$  of the spherical model in regime I with  $\alpha < -2$  is evident.

Therefore, we conjecture that the relationship between  $\lambda_C$  and  $\lambda_R$  is, in general, given by equation (4.3) and that only in those special cases where  $\alpha_{\text{eff}} = 0$  applies, these two exponents happen to be equal. That was indeed the case in all models reviewed in [4]. Tests of the conjectured scaling relation (4.3) in other models would be most welcome.

Similarly, the usual anticipation, see, e.g. [4], that at criticality the two exponents  $a = b = 2\beta/\nu z$ , where  $\beta$  and  $\nu$  are standard equilibrium critical exponents, can be checked through the entries of table 2. It appears again more as a property of certain initial conditions than as a generally valid statement. From table 2,  $a = b$  appears to hold whenever  $X(t, s)$  does not vanish identically for all  $x$  in the scaling limit. In these cases the proposed relationship with the exponents  $\beta, \nu, z$  seems rather to be a hyperscaling relation since it does not hold in regime IV.

3. Having examined the asymptotic properties of the scaling functions  $f_C(x)$  and  $f_R(x)$  for large  $x$ , we now turn to their functional form for finite values of  $x$ . Indeed, as already mentioned in the introduction, conformal invariance predicts that the scaling function of the autoresponse function should be given by [18, 19]

$$f_R(x) = x^{1+a-\lambda_R/z} (x-1)^{-1-a}. \quad (4.5)$$

That prediction can be tested by comparing with the exact result (3.14) for a critical quench and with (3.27) for a quench into the ordered phase. Inserting the values of the exponents  $a$  and  $\lambda_R$  which can be read from table 2 and from equations (3.27) and (3.28), respectively, shows perfect agreement, both below criticality and at criticality for all five ageing regimes.

In addition, we may also test the full spacetime-dependent response function. Since the dynamical exponent  $z = 2$  in our model, conformal invariance predicts for  $z = 2$  [18, 29]

$$R_r(t, s) = R(t, s) \exp\left(-\frac{\mathcal{M}}{2} \frac{r^2}{t-s}\right) \quad (4.6)$$

where spatial translation invariance is already taken into account, the autoresponse function  $R(t, s)$  is given as before by equation (1.6) and  $\mathcal{M}$  is a non-universal and dimensional constant. To check this, we calculate the full response function from equation (2.20). In the scaling limit we are interested in, where both  $t, s$  as well as their difference  $t - s$  become simultaneously large, we may use the  $q \rightarrow 0$  limit (2.6) in the integral

$$R_r(t, s) = (2\pi)^{-d} \sqrt{\frac{g(T, s)}{g(T, t)}} \int d\mathbf{q} \exp[-\omega(\mathbf{q})(t-s) - i\mathbf{q} \cdot \mathbf{r}] \quad (4.7)$$

and it is easy to see that the resulting Gaussian integrals reproduce equation (4.6) exactly, with  $\mathcal{M} = 1/2$ .

This suggests that the presence of conformal invariance in ageing systems should be independent of spatially long-ranged correlations in the initial state. Further tests of conformal invariance in different ageing systems with spatially long-range initial conditions are called for.

Summarizing the study of the influence of spatially long-range correlations in the initial state on the long-time behaviour of two-time observables of the exactly solvable spherical model has led us to the recognition of several new types of non-equilibrium critical dynamics in that model. As a consequence, we could formulate the conjecture equation (4.3) on the relationship between the autocorrelation exponent  $\lambda_C$  and the autoresponse exponent  $\lambda_R$ . Finally, the hypothesis of conformal invariance in ageing ferromagnetic systems could be tested in a new way.

### Acknowledgments

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### Appendix. Some remarks on the 1D Glauber–Ising model

Here we consider briefly some non-fully disordered initial conditions in the 1D kinetic Ising model with Glauber [30] dynamics. The Hamiltonian is on a chain of  $N$  sites with periodic boundary conditions

$$\mathcal{H} = -J \sum_{i=1}^N \sigma_i \sigma_{i+1} \quad (\text{A1})$$

where  $\sigma_i = \pm 1$  are the Ising spins. The dynamics may be given through the heat bath rule, which gives the probability of finding the spin variables  $\sigma_i(t+1)$  in terms of those at time  $t$

$$P(\sigma_i(t+1) = \pm 1) = \frac{1}{2} \left[ 1 \pm \tanh \left( \frac{J}{T} (\sigma_{i-1}(t) + \sigma_{i+1}(t)) \right) \right]. \quad (\text{A2})$$

Here we consider the following initial conditions in terms of the spin–spin correlator  $C_r(t) = \langle \sigma_r(t) \sigma_0(t) \rangle$ :

$$C_r(0) = M_0^2 + (1 - M_0^2) \delta_{r,0} \quad (\text{A3})$$

where  $M_0 = M_r(0)$  is the initial averaged magnetization. This allows a simple case study of situations where the mean magnetization  $M_r(t)$  does not vanish. Here and in the following spatial translation invariance is already taken into account. We study the long-time evolution of the 1D model after a quench to zero temperature at time  $t = 0$ .

The exact solution of the model is closely parallel to the standard lines as presented for the special case  $M_0 = 0$  in [10, 11]. We shall therefore merely state our results. First, the mean magnetization  $M_r(t) = M_r(0) = M_0$  does remain constant for all times  $t$ . We are interested in the *connected* autocorrelation function  $\Gamma(t, s)$  and autoresponse function  $R(t, s)$  defined as

$$\Gamma(t, s) = \langle \sigma_r(t) \sigma_r(s) \rangle - \langle \sigma_r(t) \rangle \langle \sigma_r(s) \rangle \quad (\text{A4})$$

$$R(t, s) = T \left. \frac{\delta M_r(t)}{\delta H_r(s)} \right|_{H_r=0} \quad (\text{A5})$$

where  $H_r$  is an external magnetic field at the site  $r$ .



In 1D, we find in the scaling limit  $t, s \rightarrow \infty$  but with  $x = t/s > 1$  fixed

$$\Gamma(t, s) = (1 - M_0^2) \frac{2}{\pi} \arctan \sqrt{\frac{2}{x-1}} \quad R(t, s) = (1 - M_0^2) \frac{1}{\pi s} \sqrt{\frac{1}{2(x-1)}} \quad (\text{A6})$$

and the fluctuation–dissipation ratio  $X(t, s) = R(t, s)(\partial\Gamma(t, s)/\partial s)^{-1}$  becomes a function of  $x$  only and reads  $X(t, s) = X(x) = (x+1)/2x$ . Up to the prefactor  $1 - M_r(t)M_r(s) = 1 - M_0^2$ , the results for  $\Gamma(t, s)$  and  $R(t, s)$  (and therefore also for  $X(x)$ ) are exactly the same as found in [10, 11] for the autocorrelation and autoresponse functions at  $M_0 = 0$ . Therefore, unless  $M_0 = 1$  and the system is prepared in a zero-temperature equilibrium state, the exponent  $\alpha$  introduced in the text takes the value  $\alpha_{\text{eff}} = 0$  here. That agrees with the fact that the autocorrelation and autoresponse exponents are equal,  $\lambda_C = \lambda_R = 1$ .

It would be interesting to study the effects of non-fully disordered initial conditions in wider settings, e.g. in higher dimensions, conserved order parameters and so on.

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